Ancestral Logic

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Many efforts have been made in recent years to construct computerized systems for mechanizing general mathematical reasoning. Most of the systems are based on logics which are stronger than first-order logic (FOL). However, there are good reasons to avoid using full second-order logic (SOL) for this task. We investigate a logic which is intermediate between FOL and SOL, and seems to be a particularly attractive alternative to both: *Ancestral Logic*. This is the logic which is obtained from FOL by augmenting it with the transitive closure operator TC.

The expressive power of Ancestral Logic is equivalent to that of some of the other known intermediate logics (such as weak second-order logic, ω-logic, etc), yet there are several reasons to prefer it over the others. One of them is that it seems like the easiest choice from a *proof-theoretic* point of view. Another important reason is simply the simplicity of the notion of transitive closure. Any person, even with no mathematical background whatsoever, can easily grasp the concept of the ancestor of a given person (or in other words, the idea of transitive closure of a certain binary relation).

We argue that the concept of transitive closure is the key for understanding finitary inductive definitions and reasoning, and we provide evidence for the thesis that logics which are based on it (in which induction is a logical rule) are the right *logical* framework for the formalization and mechanization of Mathematics. We show that with TC one can define all recursive predicates and functions from 0, the successor function and addition, yet with TC alone addition is not definable from 0 and the successor function. However, in the presence of a pairing function, TC does suffice for having all types of finitary inductive definitions of relations and functions.